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Journal of Complexity 24 (2008) 259–282

Journal of COMPLEXITY

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Relative widths of smooth functions determined by fractional order derivatives[☆]

Liu Yongping[a,*], Yang Lianhong[b]

[a] *School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, PR China*

[b] *Department of Applied Mathematics, Communication University of China, Beijing 100024, PR China*

Received 8 August 2006; accepted 11 December 2006
Available online 26 January 2007

Abstract

For two subsets W and V of a normed space X. The relative Kolmogorov n-width of W relative to V in X is defined by

$$K_n(W, V)_X := \inf_{L_n} \sup_{f \in W} \inf_{g \in V \cap L_n} \|f - g\|_X,$$

where the infimum is taken over all n-dimensional subspaces L_n of X. For $\alpha \in \mathbb{R}_+$, define W_p^α $(1 \leqslant p \leqslant \infty)$ to be the collection of 2π-periodic and continuous functions f representable as a convolution

$$f(t) = c + (B_\alpha * g)(t),$$

where $g \in L_p(T)$, $T = [0, 2\pi]$, $\|g\|_p \leqslant 1$, $\int_T g(x)dx = 0$, and $B_\alpha(t) \in L_1(T)$ with the Fourier expanded form

$$B_\alpha(t) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} (ik)^{-\alpha} e^{ikt}.$$

In this article, we discuss the relative Kolmogorov n-width of W_p^α relative to W_p^α in the space $L_q(T)$. For the case $p = \infty$, $1 \leqslant q \leqslant \infty$, and the case $p = 1$, $1 \leqslant q \leqslant 2$, $\alpha > 1 - \frac{1}{q}$ and the case $p = 1$, $2 < q \leqslant \infty$, $\alpha > 3 - \frac{1}{q}$, we obtain their weak asymptotic results. In addition, we also obtain the weak asymptotic result of W_p^α relative to W_p^α in the space $L_p(T)$ for $0 < \alpha \leqslant 2$.

Keywords: Fractional order derivatives; Width; Relative width

[☆] Supported partly by National Natural Science Foundation of China (No. 10471010) and partly by the project "Representation Theory and Related Topics" of the "985 Program" of Beijing Normal University and Beijing Natural Science Foundation (1062004).

* Corresponding author.
E-mail addresses: ypliu@bnu.edu.cn (Y. Liu), yhongl@163.com (L. Yang).

doi:10.1016/j.jco.2006.12.001

1. Preliminaries and main results

Suppose $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$ denote the set of all real numbers, all integral numbers and all positive integral numbers, respectively.

Let $X_{2\pi}$ be one of the space $C(T)$ ($T = [0, 2\pi]$) of the 2π-periodic continuous functions on $\mathbb{R}$ or the space $L_p(T)$ ($1 \leqslant p \leqslant \infty$) of the 2π-periodic functions on $\mathbb{R}$ of the p-power integral functions on T, with norms

$$\|f\|_{C(T)} = \sup_{x\in T} |f(x)|, \quad \|f\|_p = \left\{\int_T |f(x)|^p\,dx\right\}^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in T} |f(x)|, \quad p = \infty,$$

respectively.

For $k \in \mathbb{Z}$, the kth Fourier coefficient of $f \in X_{2\pi}$ is denoted by

$$\hat{f}(k) = \frac{1}{2\pi}\int_T f(u)e^{-iku}\,du,$$

and the convolution of $f \in L_1(T)$, $g \in X_{2\pi}$ by

$$(f * g)(x) = \int_T f(x-u)g(u)\,du.$$

To give our results, we need the concepts of the derivative, the integral and the modulus of continuity of fractional order $\alpha > 0$ (see [5,7]) as follows.

The (right) difference of $f \in X_{2\pi}$ of fractional order $\alpha > 0$ with respect to the increment $h \in \mathbb{R}$ is defined by

$$(\Delta_h^\alpha f)(x) = \sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f(x-hj) \quad (x \in \mathbb{R}),$$

where

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}, \quad j = 0, 1, 2, \ldots,$$

which coincides with $\Delta_h^r f(x)$ for $\alpha = r$, $r \in \mathbb{Z}_+$ (noting that $\binom{r}{j} = 0$ for $j \geqslant r+1$). If for $f \in L_p(T)$ there exists $g \in L_p(T)$ such that

$$\lim_{h\to 0+}\left\|\frac{\Delta_h^\alpha f}{h^\alpha} - g\right\|_p = 0, \quad 1 \leqslant p < \infty. \tag{1.1}$$

Then g will be called the αth Liouville–Grünwald derivative of f in the mean of order p, denoting $g = f^{(\alpha)}$. If f and g are functions belonging to $C(T)$ and if (1.1) holds with the $L_p(T)$-norm replaced by the $C(T)$-norm, then we will speak of $f^{(\alpha)}$ as the αth uniform Liouville–Grünwald derivative of f. For simplicity we will write briefly $f^{(\alpha)} \in L_p(T)$ or $f^{(\alpha)} \in C(T)$.

The modulus of continuity of $f \in X_{2\pi}$ of index $\alpha > 0$ is defined by

$$w_\alpha(f, \delta, X_{2\pi}) = w_\alpha(f, \delta) = \sup_{0<h\leqslant\delta} \|\Delta_h^\alpha f\|_{X_{2\pi}} \quad (\delta > 0).$$

For $\alpha \in \mathbb{R}_+$, denote by $W_p^\alpha\,(1\leqslant p\leqslant\infty)$ to be the collection of 2π-periodic and continuous functions f representable as a convolution

$$f(t) = c + (B_\alpha * g)(t),$$

where $g \in L_p(T)$, $\|g\|_p\leqslant 1$, $\int_T g(x)\,dx = 0$, and $B_\alpha(t) \in L_1(T)$ with the Fourier expanded form

$$B_\alpha(t) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}\backslash\{0\}} (ik)^{-\alpha} e^{ikt}$$

(see [31]). We also call that f is the indefinite integral of g of fractional order $\alpha > 0$.

It is not hard to verify the following relations: for $1\leqslant p < \infty$, g is the αth Liouville–Grühwald derivative of f in the mean of order p. For $p = \infty$, if $g \in C(T)$, then it is the αth uniform Liouville–Grünwald derivative of f. In the view of [6, p. 172], the function class $W_p^\alpha(T)$ coincides with the usual Sobolev class $W_p^r(T)$ for $\alpha = r$, $r \in \mathbb{Z}_+$. So, here and in what follows, we will also call g is the αth fractional derivative of f, denoting $g = f^{(\alpha)}$. On the basic properties of the fractional difference and derivative may be seen in [5,7].

Konovalov in [11] introduced the notion of the relative widths. Let X be a normed space, W and V are two centrally symmetric subsets of X. The relative Kolmogorov n-width of W relative to V in X is defined by

$$K_n(W, V)_X := \inf_{L_n} \sup_{f\in W} \inf_{g\in V\cap L_n} \| f - g \|_X, \tag{1.2}$$

where the infimum is taken over all n-dimensional subspaces L_n of X. When $V = X$, the relative Kolmogorov n-width of W relative to V in X coincides with the usual Kolmogorov n-width $d_n(W, X)$ of W in X. Obviously, $K_n(W, V, X)\geqslant d_n(W, X)$ for any set $V \subset X$.

Kolmogorov [9] established that $d_{2n-1}(W_2^r, L_2(T)) = d_{2n}(W_2^r, L_2(T)) = n^{-r}$. The exact order $d_n(W_\infty^r, L_\infty(T)) \asymp n^{-r}$ of Kolmogorov widths was determined by Stechkin in [20]. Here and further, the notation $a_n \ll b_n$ (or $b_n \gg a_n$) denotes that there is a constant $c > 0$ such that $|a_n|\leqslant c|b_n|$ for all $n \in \mathbb{Z}_+$. The notation $a_n \asymp b_n$ denotes that both $a_n \ll b_n$ and $a_n \gg b_n$ hold.

For $n, r \in \mathbb{Z}_+$, Tikhomirov [26,27] determined the values of the widths

$$d_{2n-1}(W_\infty^r, L_\infty(T)) = d_{2n}(W_\infty^r, L_\infty(T)) = \kappa_r n^{-r},$$

where κ_r are the well-known Favard constants. In Korneichuk's book [16], we also see that

$$d_{2n-1}(W_1^r, L_1(T)) = d_{2n}(W_1^r, L_1(T)) = \kappa_r n^{-r}.$$

On the relative Kolmogorov n-width, for each $r \in \mathbb{Z}_+$, Konovalov in [11] first obtained the following results

$$K_n(W_\infty^r, W_\infty^r)_\infty \asymp n^{-2}, \quad r\geqslant 3.$$

Babenko in [1] proved that,

$$K_n(W_1^r, W_1^r)_1 \asymp n^{-2}, \quad r\geqslant 3.$$

The indicated difference in the behavior of the Kolmogorov widths and relative widths aroused a certain interest in the problem of the behavior of the widths $K_n(W_p^r, MW_p^r)_q$.

In [12,13], Konovalov established that the following relations hold: for each $r \in \mathbb{Z}_+$ and $1 \leqslant q \leqslant \infty$,

$$K_n(W_2^r, W_2^r)_q \asymp n^{-\min\{r-\frac{1}{2}+\frac{1}{q},\ r\}},$$

$$K_n(W_\infty^r, W_\infty^r)_q \asymp n^{-\min\{r,\ 2\}},$$

$$K_n(W_1^r, W_1^r)_q \asymp n^{-\min\{r-1+\frac{1}{q},\ 2\}}, \quad (r, q) \neq (1, \infty).$$

Tikhomirov [29] considered the relative widths of the classes W_∞^α for non-integer $\alpha > 0$, and proved that

$$K_n(W_\infty^\alpha, W_\infty^\alpha)_\infty \asymp n^{-\min\{\alpha,\ 2\}}, \quad \alpha > 0.$$

From the above-mentioned results, we will see that it is interesting to compare the two quantities $K_n(W_p^r, MW_p^r)_q$ and $d_n(W_p^r, L_q)$, for some constant $M > 0$, and to find the smallest number M which make the following equality $K_n(W_p^r, MW_p^r)_q = d_n(W_p^r, L_q)$. On the aspect's study, readers may refer to [2,3,22,23]. In the case of many variables, readers may refer to Babenko's paper [1]. In [30,17], we also generalized some results in [22,12] from univariate to many variables. Recently, Konovalov and Leviatan also study the problems of so called shape-preserving widths of some Sobolev type classes by taking V as the set of s-monotone functions on the interval T in (1.2) and get a large and serious of works (for example, [14,15,8]).

In the present paper, we continue studying the relative widths $K_n(W_p^\alpha, W_p^\alpha)_q$ for non-integer $\alpha > 0$, and obtain the following main results.

Theorem 1. *For $\alpha \in \mathbb{R}_+$, $0 < \alpha \leqslant 2$ and $1 \leqslant p \leqslant \infty$. Then*

$$n^{-\alpha} \ll d_n(W_p^\alpha, L_p(T)) \leqslant K_n(W_p^\alpha, W_p^\alpha)_p \ll n^{-\alpha}, \quad n \in \mathbb{Z}_+.$$

When $p = \infty$, Theorem 1 may be seen in Tikhomirov [29].

Theorem 2. *Let $\alpha \in \mathbb{R}_+$ and $1 \leqslant q < \infty$. Then*

$$K_n(W_\infty^\alpha, W_\infty^\alpha)_q \asymp n^{-\min\{\alpha,\ 2\}}, \quad n \in \mathbb{Z}_+. \tag{1.3}$$

Theorem 3. *Let $\alpha \in \mathbb{R}_+$ and $n \in \mathbb{Z}_+$. Then*

$$K_n(W_1^\alpha, W_1^\alpha)_q \begin{cases} \asymp n^{-2}, & \alpha \geqslant 3 - \dfrac{1}{q}, \ 1 \leqslant q \leqslant \infty, \\ \asymp n^{-(\alpha-1+\frac{1}{q})}, & 1 - \dfrac{1}{q} < \alpha < 3 - \dfrac{1}{q}, \ 1 \leqslant q \leqslant 2, \\ \ll n^{-(\alpha-1+\frac{1}{q})}, & 1 - \dfrac{1}{q} < \alpha < 3 - \dfrac{1}{q}, \ 2 < q \leqslant \infty. \end{cases} \tag{1.4}$$

Remark. Theorem 3 shows that

$$K_n(W_1^\alpha, W_1^\alpha)_q \asymp n^{-\min\{\alpha-1+\frac{1}{q},\ 2\}}, \quad n \in \mathbb{Z}_+,$$

for the case $\alpha > 1 - \frac{1}{q}$, $1 \leqslant q \leqslant 2$.

2. Preliminary lemmas

In the proofs of Theorem 2 and 3, we shall also use the following facts.
Here and in what follows, let

$$E(W, M)_X = \sup_{f \in W} \inf_{g \in M} \|f - g\|_X$$

denotes the best approximation of a set W by a set $M \subset X$.

Below, we present the known duality theorem about best Approximation in the space L_p (see, e.g. [16]).

Lemma 1. *Let $F \subset L_p[a, b]$ $(1 \leqslant p \leqslant \infty)$ be a convex and closed subset. Then for any $f(t) \in L_p[a, b]$, and $\frac{1}{p} + \frac{1}{p'} = 1$, we have*

$$E(f, F)_{L_p[a,b]} = \sup_{\|g\|_{L_{p'}[a,b]} \leqslant 1} \left\{ \int_a^b f(t) g(t)\, dt - \sup_{u \in F} \int_a^b u(t) g(t)\, dt \right\}.$$

In the paper, we also will use the following Fourier expansion formula:

$$\varphi_0(t) := \operatorname{sgn} \sin t = \frac{4}{\pi} \sum_{k=1}^{\infty} (2k-1)^{-1} \sin(2k-1)t.$$

For each $r \in \mathbb{Z}_+$, denote by $\varphi_r(t)$ the 2π-periodic perfect Euler spline of order r which its r-derivative is $\varphi_r^{(r)}(t) = \varphi_0(t)$, $t \neq k\pi$, $\forall k \in \mathbb{Z}$.

From [18, p. 416], we see the following fact:

Lemma 2. *Let $w \in L_2(T)$ be some fixed function with mean value zero, and its Fourier series can be expressed as follows*:

$$w(t) = \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt).$$

Denote by $k(w)$ the subset of $L_2(T)$ formed by the translates of $w(\cdot)$, that is

$$k(w) = \{T_\tau w(\cdot)\}_{\tau \in T}, \quad T_\tau w(t) = w(t + \tau).$$

Then, for $n = 1, 2, \ldots$,

$$d_{2n}(k(w), L_2(T)) = \left(\pi \sum_{k=n+1}^{\infty} c_k^{*2} \right)^{\frac{1}{2}},$$

where c_k^* denote the numbers $c_k = (a_k^2 + b_k^2)^{\frac{1}{2}}$ arranged in non-increasing order.

Lemma 3. *Let $y(\cdot) \in C(\triangle)$, $y(t) \geqslant 0$, for any $t \in \triangle = [t_0, t_1]$,*

$$X = \{x(\cdot) \in L_1(\triangle) \colon 0 \leqslant x(t) \leqslant A,\ \text{a.e.},\ \int_\triangle x(t)\, dt \geqslant B\}.$$

Then

$$\int_{\triangle} x(t)y(t)\,dt \geqslant A \int_{D(A,B)} y(t)\,dt, \quad \forall x(\cdot) \in X,$$

where $D(A, B) = \{t \mid 0 \leqslant y(t) \leqslant C(A, B)\}$, *while the constant* $C(A, B)$ *is chosen so as to have* $\int_{D(A,B)} dt = \frac{B}{A}$.

Lemma 3 may be found in [29]. To be easily read we also give its proof as follows.

Proof.

$$\begin{aligned}
&\int_{\triangle} x(t)y(t)\,dt - A\int_{D(A,B)} y(t)\,dt \\
&\quad = \int_{D(A,B)} y(t)(x(t) - A)\,dt + \int_{\triangle\setminus D(A,B)} x(t)y(t)\,dt \\
&\quad \geqslant C(A, B)\int_{\triangle\setminus D(A,B)} x(t)\,dt - C(A, B)\int_{D(A,B)} (A - x(t))\,dt \\
&\quad = C(A, B)\int_{\triangle} x(t)\,dt - C(A, B)\int_{D(A,B)} A\,dt \\
&\quad = C(A, B)\int_{\triangle} x(t)\,dt - C(A, B)B \geqslant 0. \qquad \square
\end{aligned}$$

3. Kolmogorov type inequality

To obtain the lower bound of the widths $K_n(W_\infty^\alpha, W_\infty^\alpha)_q$ for $0 < \alpha < 2$ in Theorem 2, we first need to prove the following Kolmogorov type inequality for fractional derivatives.

It is well-known that Kolmogorov has given the following inequality [10].

3.1. *Kolmogorov inequality*

For each function *f* which satisfies that $f^{(r-1)}$ is locally absolutely continuous on $\mathbb{R}$ and $f^{(r)} \in L_\infty(\mathbb{R})$. Then

$$\|f^{(k)}\|_{L_\infty(\mathbb{R})} \leqslant C_{k,r} \|f\|_{L_\infty(\mathbb{R})}^{1-\frac{k}{r}} \|f^{(r)}\|_{L_\infty(\mathbb{R})}^{\frac{k}{r}}, \tag{3.1}$$

where $k,\ r \in \mathbb{Z}_+,\ 0 < k < r$, and $C_{k,r} = (K_{r-k})/(K_r)^{1-\frac{k}{r}}$, while

$$K_i = \|\varphi_i\|_{C(T)} = \begin{cases} \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty} (-1)^k/(2k+1)^{i+1}, & i \text{ even}, \\[2ex] \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty} 1/(2k+1)^{i+1}, & i \text{ odd}. \end{cases}$$

Here $\varphi_r(t)$ is the 2π-periodic perfect Euler spline of order r defined in the last section. Moreover, the inequalities are best possible, since there exist piecewise smooth functions for which equality is attained.

Stein in [21] generalized the Kolmogorov's inequality to the space L_p and obtained the following results. Let f be a continuous function satisfying that $f,\ f' \cdots f^{(r-1)}$ are locally absolutely continuous, and $f,\ f' \cdots f^{(r)}$ all belong to $L_p(\mathbb{R})$ for some $p,\ 1 \leqslant p < \infty$. Then

$$\|f^{(k)}\|_{L_p(\mathbb{R})} \leqslant C_{k,r} \|f\|_{L_p(\mathbb{R})}^{1-\frac{k}{r}} \|f^{(r)}\|_{L_p(\mathbb{R})}^{\frac{k}{r}},$$

where $C_{k,r}$ are the same as in (3.1).

Babenko in [4] generalized inequality (3.1) to the case of fractional derivatives and obtained the following result that if a function f satisfies the conditions in (3.1), then there holds

$$\|D^\alpha f\|_{L_\infty(\mathbb{R})} \leqslant \frac{\|D^\alpha \tilde{\varphi}_r\|_{L_\infty(\mathbb{R})}}{\|\tilde{\varphi}_r\|_{L_\infty(\mathbb{R})}} \|f\|_{L_\infty(\mathbb{R})}^{1-\frac{\alpha}{r}} \|f^{(r)}\|_{L_\infty(\mathbb{R})}^{\frac{\alpha}{r}}, \tag{3.2}$$

where $0 < \alpha < 1$ or $1 < \alpha < 2,\ r = 2, 3, \ldots,$ while

$$(D^\alpha f)(x) = \int_0^\infty \frac{f(x) - f(x-t)}{t^{1+\alpha}}\, dt \quad \text{if } 0 < \alpha < 1, \tag{3.3}$$

$$(D^\alpha f)(x) = \int_0^\infty \frac{\Delta_h^2 f(x)}{h^{1+\alpha}}\, dh \quad \text{if } 1 < \alpha < 2,$$

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h),$$

and

$$\tilde{\varphi}_r(x) = \begin{cases} \varphi_r(x + (r-1)\frac{\pi}{2}), & x \in [-\pi, \pi], \\ \max\limits_x \varphi_r(x), & x \notin [-\pi, \pi]. \end{cases}$$

By [7,31], it is easy to verify that there exists a constant c_α such that

$$D^\alpha f = c_\alpha f^{(\alpha)}. \tag{3.4}$$

From this fact and combining relation (3.2), we obtain

$$\|f^{(\alpha)}\|_{L_\infty(\mathbb{R})} \leqslant \frac{\|\tilde{\varphi}_r^{(\alpha)}\|_{L_\infty(\mathbb{R})}}{\|\tilde{\varphi}_r\|_{L_\infty(\mathbb{R})}} \|f\|_{L_\infty(\mathbb{R})}^{1-\frac{\alpha}{r}} \|f^{(r)}\|_{L_\infty(\mathbb{R})}^{\frac{\alpha}{r}}, \tag{3.5}$$

for $0 < \alpha < 1$, and $1 < \alpha < 2$.

For any $\alpha \in \mathbb{R}_+,\ \alpha \notin \mathbb{Z}_+,\ \alpha < r$, set $\alpha = [\alpha] + \lambda,\ 0 < \lambda < 1$, and set $g = f^{([\alpha])}$. Here $[\alpha]$ denotes the biggest integral number less than α. By using inequality (3.5), we have

$$\|g^{(\lambda)}\|_{L_\infty(\mathbb{R})} \leqslant \frac{\|\tilde{\varphi}_{r-[\alpha]}^{(\lambda)}\|_{L_\infty(\mathbb{R})}}{\|\tilde{\varphi}_{r-[\alpha]}\|_{L_\infty(\mathbb{R})}} \|g\|_{L_\infty(\mathbb{R})}^{1-\frac{\lambda}{r-[\alpha]}} \|g^{(r-[\alpha])}\|_{L_\infty(\mathbb{R})}^{\frac{\lambda}{r-[\alpha]}},$$

while using inequality (3.1), we have

$$\|f^{([\alpha])}\|_{L_\infty(\mathbb{R})} \leqslant C_{[\alpha],r}\|f\|_{L_\infty(\mathbb{R})}^{1-\frac{[\alpha]}{r}}\|f^{(r)}\|_{L_\infty(\mathbb{R})}^{\frac{[\alpha]}{r}}.$$

Combining the above two inequalities, we deduce

$$\|f^{(\alpha)}\|_{L_\infty(\mathbb{R})} \leqslant \frac{\|\tilde{\varphi}_{r-[\alpha]}^{(\lambda)}\|_{L_\infty(\mathbb{R})}}{\|\tilde{\varphi}_{r-[\alpha]}\|_{L_\infty(\mathbb{R})}} C_{[\alpha],r}^{(1-\frac{\lambda}{r-[\alpha]})}\|f\|_{L_\infty(\mathbb{R})}^{1-\frac{\alpha}{r}}\|f^{(r)}\|_{L_\infty(\mathbb{R})}^{\frac{\alpha}{r}},$$

that is

$$\|f^{(\alpha)}\|_{L_\infty(\mathbb{R})} \leqslant C\|f\|_{L_\infty(\mathbb{R})}^{1-\frac{\alpha}{r}}\|f^{(r)}\|_{L_\infty(\mathbb{R})}^{\frac{\alpha}{r}}, \quad 0 < \alpha < r, \tag{3.6}$$

with an absolute constant C.

Using the similar way as in [21], we may prove the following theorem.

Theorem 4. *Suppose that the Liouville–Grünwald derivative $f^{(\alpha)}$ exists as an element in $L_p(\mathbb{R})$, $1 \leqslant p < \infty$ and assume further that f and $f^{(r)}$ belong to $L_p(\mathbb{R})$. Then*

$$\|f^{(\alpha)}\|_{L_p(\mathbb{R})} \leqslant C\|f\|_{L_p(\mathbb{R})}^{1-\frac{\alpha}{r}}\|f^{(r)}\|_{L_p(\mathbb{R})}^{\frac{\alpha}{r}}, \quad 0 < \alpha < r, \tag{3.7}$$

where C only depends on α and r.

To limit the long of the paper, we omit the proof of Theorem 4.

Remark. Theorem 4 is also true in the space $L_p(T)$. It can be proved similarly in view of the fact that the inequality (3.2) is true in the space $L_p(T)$.

4. Proof of Theorem 1

4.1. Upper estimate

Let

$$(J_n f)(x) = \int_T f(x+t)k_n(t)\,dt, \tag{4.1}$$

where $k_n(t) = L_{n'}(t)$ $(n' = [\frac{n}{2}] + 1)$, while

$$L_n(t) = \lambda_n^{-1}\left(\frac{\sin\dfrac{nt}{2}}{\sin\dfrac{t}{2}}\right)^4, \quad \int_T L_n(t)\,dt = 1.$$

Here λ_n is determined by the last equality. It is well-known that $J_n(f)$ is a trigonometric polynomial of degree not greater than n.

If $f \in W_p^\alpha$, $0 < \alpha \leqslant 2$, it is easy to see that

$$(J_n f)^{(\alpha)}(x) = \int_T f^{(\alpha)}(x+t)k_n(t)\,dt$$

from (4.1). Then, by the generalized Minkowski inequality we have

$$\|(J_n f)^{(\alpha)}\|_p = \left(\int_T \left|\int_T f^{(\alpha)}(x+t)k_n(t)\,dt\right|^p dx\right)^{\frac{1}{p}}$$
$$\leqslant \int_T \left(\int_T \left|f^{(\alpha)}(x+t)k_n(t)\right|^p dx\right)^{\frac{1}{p}} dt$$
$$\leqslant \|f^{(\alpha)}\|_p \int_T |k_n(t)|\,dt \leqslant 1,$$

that is $J_n f \in W_p^\alpha$.

In what follows, we estimate the deviation

$$\|f - J_n f\|_p.$$

By the property of the fractional difference (see [7]), we may see that

$$\Delta_t^{\alpha+\beta} f(x) = \Delta_t^\alpha(\Delta_t^\beta f)(x)$$

for all $f \in L_p(T)$, and the positive real numbers α, β, and that

$$\|\Delta_t^\alpha f\|_p = O(|t|^\alpha) \quad \text{as } t \to 0,$$

for each $f \in W^\alpha(T)$. Notice that

$$\int_0^\pi t^\alpha k_n(t)\,dt \asymp n^{-\alpha}, \quad n \in \mathbb{Z}_+,$$

for all $0 < \alpha \leqslant 2$. Thus, we can derive

$$\|\Delta_t^2 f\|_p = \|\Delta_t^{2-\alpha}(\Delta_t^\alpha f)\|_p \leqslant 2^{\{2-\alpha\}} \|\Delta_t^\alpha f\|_p,$$

and hence

$$\|f - J_n f\|_p = \left\|\int_T [f(x) - f(x+t)]k_n(t)\,dt\right\|_p$$
$$= \left\|\int_0^\pi \Delta_t^2 f(x)k_n(t)\,dt\right\|_p \leqslant 2^{\{2-\alpha\}} \int_0^\pi \|\Delta_t^\alpha f(x)\|_p k_n(t)\,dt$$
$$\leqslant 2^{\{2-\alpha\}} c(\alpha)\|f^{(\alpha)}\|_p \int_0^\pi t^\alpha k_n(t)\,dt \leqslant c'(\alpha)\|f^{(\alpha)}\|_p n^{-\alpha} \leqslant c'(\alpha) n^{-\alpha},$$

where $c'(\alpha)$ is a constant which depends only on α.

Thus, we complete the upper estimate of the asymptotic formula in Theorem 1.

4.2. Lower estimate

Set

$$A_n = \left\{T : T = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx),\ \|T\|_p \leqslant n^{-\alpha}\right\}.$$

By the Bernstein–Nikol'skii inequalities (see [28])

$$\|T^{(\alpha)}\|_p \leqslant cn^\alpha \|T\|_p.$$

It follows $A_n \subset cW_p^\alpha$.

By the well-known theorem about the width of a ball, we have

$$d_n(W_p^\alpha, L_p) \geqslant d_n\left(\frac{1}{c}A_n, L_p\right) \geqslant \frac{1}{c}n^{-\alpha}.$$

Hence,

$$K_n(W_p^\alpha, W_p^\alpha)_p \geqslant d_n(W_p^\alpha, L_p) \gg n^{-\alpha}.$$

Thus, the lower estimate is finished.

5. Proof of Theorem 2

Since

$$K_n(W_\infty^\alpha, W_\infty^\alpha)_q \leqslant (2\pi)^{\frac{1}{q}} K_n(W_\infty^\alpha, W_\infty^\alpha)_\infty,$$

the upper bound in (1.3) is a consequence of the upper bound for $K_n(W_\infty^\alpha, W_\infty^\alpha)_\infty$ given in [29]. Therefore, it remains to prove the desired lower bound.

5.1. Lower estimate

The case $0 < \alpha < 2$. Since

$$K_n(W_\infty^\alpha, W_\infty^\alpha)_q \geqslant d_n(W_\infty^\alpha, L_q(T)) \geqslant (2\pi)^{\frac{1}{q}-1} d_n(W_\infty^\alpha, L_1(T)), \quad 1 \leqslant q \leqslant \infty,$$

all that we need to do is to prove the lower estimate of $d_n(W_\infty^\alpha, L_1(T))$, it follows from a discretization method.

As seen in Pinkus's book [19], let ϕ be any non-zero function in $C^\infty(\mathbb{R})$ with support in $[0, 2\pi]$, i.e., $\mathrm{supp}(\phi) \subset [0, 2\pi]$, and set $\phi_k(x) = \phi(xM - 2\pi(k-1)),\ k = 1, \ldots, M$. Thus

$$\mathrm{supp}(\phi_k) \subset \left[\frac{2\pi(k-1)}{M}, \frac{2\pi k}{M}\right], \quad k = 1, \ldots, M.$$

For every $s \in [1, \infty]$, a simple change of variable argument shows that

$$\|\phi_k\|_s = M^{-\frac{1}{s}} \|\phi\|_s.$$

Furthermore, since the ϕ_k have essentially distinct support,

$$\left\|\sum_{k=1}^M a_k \phi_k\right\|_s = M^{-\frac{1}{s}} \|a\|_s \|\phi\|_s,$$

where $a = (a_1, \ldots, a_M)$ and $\|a\|_s$ is the usual l_s^M-norm on a, $\|\phi\|_s$ is the usual L_s-norm on the function ϕ. (We are abusing notation by using $\|\cdot\|_s$ for both the $L_s[0, 2\pi]$ and the l_s^M-norm).

Similarly, for $r \in \mathbb{Z}_+$,

$$\left\| \sum_{k=1}^{M} a_k \phi_k^{(r)} \right\|_s = M^{r-\frac{1}{s}} \|a\|_s \|\phi^{(r)}\|_s.$$

Suppose that $f = \sum_{k=1}^{M} a_k \phi_k$. Then for any $\alpha > 0,\ \alpha \notin \mathbb{Z}_+,\ f^{(\alpha)} = \sum_{k=1}^{M} a_k \phi_k^{(\alpha)}$. By Kolmogorov type inequality for fractional derivatives, we have

$$\begin{aligned}
\|f^{(\alpha)}\|_s &\leqslant C\|f\|_s^{1-\frac{\alpha}{1+[\alpha]}} \|f^{(1+[\alpha])}\|_s^{\frac{\alpha}{1+[\alpha]}} \\
&\leqslant C \left\| \sum_{k=1}^{M} a_k \phi_k \right\|_s^{1-\frac{\alpha}{1+[\alpha]}} \left\| \sum_{k=1}^{M} a_k \phi_k^{(1+[\alpha])} \right\|_s^{\frac{\alpha}{1+[\alpha]}} \\
&= C(M^{-\frac{1}{s}} \|a\|_s \|\phi\|_s)^{1-\frac{\alpha}{1+[\alpha]}} (M^{1+[\alpha]-\frac{1}{s}} \|a\|_s \|\phi^{(1+[\alpha])}\|_s)^{\frac{\alpha}{1+[\alpha]}} \\
&= CM^{\alpha-\frac{1}{s}} \|a\|_s \|\phi\|_s^{1-\frac{\alpha}{1+[\alpha]}} \|\phi^{(1+[\alpha])}\|_s^{\frac{\alpha}{1+[\alpha]}}.
\end{aligned} \tag{5.0}$$

Now

$$\begin{aligned}
d_n(W_\infty^\alpha, L_1(T)) &= \inf_{X_n} \sup_{\|f^{(\alpha)}\|_\infty \leqslant 1} \inf_{g\in X_n} \|f-g\|_1 \\
&= \inf_{X_n} \sup_{\|f^{(\alpha)}\|_\infty \leqslant 1} \sup_{\substack{h\perp X_n \\ \|h\|_\infty \leqslant 1}} (f,h).
\end{aligned}$$

Here X_n is taken over all subspaces of $L_1(T)$ and $h \perp X_n$ means that $(g,h) = \int_0^{2\pi} g(x)h(x)\,dx = 0$ for all $g \in X_n$. Set $Y_M = \text{span}(\phi_1, \ldots, \phi_M)$. Thus

$$d_n(W_\infty^\alpha, L_1(T)) \geqslant \inf_{X_n} \sup_{\substack{\|f^{(\alpha)}\|_\infty \leqslant 1 \\ f\in Y_M}} \sup_{\substack{h\perp X_n \\ \|h\|_\infty \leqslant 1 \\ h\in Y_M}} (f,h).$$

Let $f = \sum_{k=1}^{M} a_k \phi_k$ and $h = \sum_{k=1}^{M} b_k \phi_k$. Then we have

$$\begin{aligned}
&\|f^{(\alpha)}\|_\infty \leqslant CM^\alpha \|a\|_\infty \|\phi\|_\infty^{1-\frac{\alpha}{1+[\alpha]}} \|\phi^{(1+[\alpha])}\|_\infty^{\frac{\alpha}{1+[\alpha]}}, \\
&\|h\|_\infty = \|\phi\|_\infty \|b\|_\infty,
\end{aligned}$$

and

$$(f,h) = \sum_{k=1}^{M} a_k b_k \|\phi_k\|_2^2 = M^{-1} \|\phi\|_2^2 \sum_{k=1}^{M} a_k b_k.$$

Assume $X_n = \text{span}\{g_1, \ldots, g_n\}$ and define $d_{i,j} = (g_i, \phi_j),\ i = 1, \ldots, n;\ j = 1, \ldots, M$. Let $d^i,\ i = 1, \ldots, n$, denote the ith row of the matrix $D = (d_{i,j})$. Then there is a constant C such that there holds the following estimate:

$$\begin{aligned}
d_n(W^\alpha_\infty, L_1(T)) &\geqslant CM^{-\alpha-1} \inf_{d^1,\dots,d^n} \sup_{\|a\|_\infty \leqslant 1} \sup_{\substack{(d^i,b)=0, i=1,\dots,n\\ \|b\|_\infty \leqslant 1}} (a,b)\\
&\geqslant CM^{-\alpha-1} \inf_{d^1,\dots,d^n} \sup_{\|a\|_\infty \leqslant 1} \inf_{\alpha_1,\dots,\alpha_n} \left\| a - \sum_{i=1}^n \alpha_i d^i \right\|_1\\
&\geqslant CM^{-\alpha-1} d_n(\mathcal{F}_\infty, l_1^M),
\end{aligned}$$

where $\mathcal{F}_\infty = \{(a_1,\dots,a_n) \colon \sup_j |a_j| \leqslant 1\}$.

See Pinkus [19, Chapter VI, Theorem 2.2], there holds

$$d_n(\mathcal{F}_\infty, l_1^M) = M - n.$$

Taking, $M = 2n$, then we get

$$d_n(W^\alpha_\infty, L_1(T)) \geqslant n^{-\alpha}.$$

The case $\alpha \geqslant 2$. It is well known that for each $f \in W^\alpha_p$, it can be expressed as

$$f(t) = \frac{1}{2\pi} \int_T f(\tau)\, d\tau + (B_\alpha * f^{(\alpha)})(t),$$

where $(B_\alpha * f^{(\alpha)})(t) := \int_T B_\alpha(t-\tau) f^{(\alpha)}(\tau)\, d\tau$ is the convolution of the derivative $f^{(\alpha)}$ with the kernel

$$B_\alpha(t) = \frac{1}{2\pi} {\sum_{k\in\mathbb{Z}}}' (ik)^{-\alpha} e^{ikt}.$$

Set $\varphi_0(t) := \operatorname{sgn} \sin t$ and $\varphi_\alpha(t) := (B_\alpha * \varphi_0)(t),\ t \in \mathbb{R}$. Denote by

$$k(\varphi_\alpha) := \{\varphi_{\alpha,\tau} : \varphi_{\alpha,\tau}(\cdot) := \varphi_\alpha(\cdot + \tau),\ \tau \in [0, 2\pi]\}$$

the curve generated by translations of the functions $\varphi_\alpha(\cdot)$. When $\alpha = r \in \mathbb{Z}_+$, $\varphi_r(t)$ is the usual 2π-periodic perfect Euler spline of order r. Obviously, $k(\varphi_\alpha) \subset W^\alpha_\infty$ and hence $K_n(W^\alpha_\infty, W^\alpha_\infty)_q \geqslant K_n(k(\varphi_\alpha), W^\alpha_\infty)_q$.

It is sufficient to prove that

$$K_n(k(\varphi_\alpha), W^\alpha_\infty)_q \geqslant n^{-2}, \quad \alpha \geqslant 2, \quad n \in \mathbb{Z}_+. \tag{5.1}$$

We fix an arbitrary subspace $L^{2n} \subset L_q$ of dimension $\leqslant 2n$ such that $L^{2n} \cap W^\alpha_\infty \neq \emptyset$. Then, by Lemma 1, we find that for every $\tau \in [0, 2\pi]$, the function $\varphi_{\alpha,\tau}(\cdot) := \varphi_\alpha(\cdot + \tau)$ satisfies the inequality

$$E(\varphi_{\alpha,\tau}, L^{2n} \cap W^\alpha_\infty)_q = \sup_{\substack{g\in L_{q'}\\ \|g\|_{q'} \leqslant 1}} \left(\int_T \varphi_{\alpha,\tau}(t) g(t)\, dt - \sup_{f \in L^{2n}\cap W^\alpha_\infty} \int_T f(t) g(t)\, dt \right),$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Setting

$$g_{q,\tau}(t) = (2\pi)^{-\frac{1}{q'}} e^{-i\alpha\pi} \varepsilon_0 \varphi_0 \left(t + \tau + \frac{\pi}{2} + \beta\pi \right),$$

here β satisfies $0 \leqslant \beta < 1$, $H(\beta\pi) = 0$, while

$$H(t) = \sum_{v=0}^{\infty} \frac{\cos[(2v+1)t - \frac{\pi\alpha}{2}]}{(2v+1)^{\alpha}}.$$

From Sun's papers [24,25], we see that $\operatorname{sgn} H(t) = \varepsilon_0 \operatorname{sgn} \sin(t - \beta\pi),\ \varepsilon_0 = \pm 1$.

Note that $g_{q,\tau} \in L_{q'}$ and $\|g_{q,\tau}\|_{q'} = 1$. Therefore, we have the relations

$$\begin{aligned} E(\varphi_{\alpha,\tau}, L^{2n} \cap W_{\infty}^{\alpha})_q &\geqslant \int_T \varphi_{\alpha,\tau}(t) g_{q,\tau}(t)\,dt - \sup_{f \in L^{2n} \cap W_{\infty}^{\alpha}} \int_T f(t) g_{q,\tau}(t)\,dt \\ &= \inf_{f \in L^{2n} \cap W_{\infty}^{\alpha}} \left(\int_T \varphi_{\alpha,\tau}(t) g_{q,\tau}(t)\,dt - \int_T f(t) g_{q,\tau}(t)\,dt \right). \end{aligned} \tag{5.2}$$

Since $B_\alpha(-x) = e^{-i\alpha\pi} B_\alpha(x)$, we can obtain the following relations

$$\begin{aligned} &\int_T \varphi_{\alpha,\tau}(t) g_{q,\tau}(t)\,dt \\ &= (2\pi)^{-\frac{1}{q'}} \int_T \int_T B_\alpha(t+\tau-u)\varphi_0(u)\,du e^{-i\alpha\pi} \varepsilon_0 \varphi_0\left(t+\tau+\frac{\pi}{2}+\beta\pi\right) dt \\ &= (2\pi)^{-\frac{1}{q'}} \int_T \int_T e^{-i\alpha\pi} B_\alpha(t+\tau-u) \varepsilon_0 \varphi_0\left(t+\tau+\frac{\pi}{2}+\beta\pi\right) dt \varphi_0(u)\,du \\ &= (2\pi)^{-\frac{1}{q'}} \int_T \int_T e^{-i\alpha\pi} B_\alpha(t-u) \varepsilon_0 \varphi_0\left(t+\tau+\frac{\pi}{2}+\beta\pi\right) dt \varphi_0(u+\tau)\,du \\ &= (2\pi)^{-\frac{1}{q'}} \int_T \int_T B_\alpha(u-t) \varepsilon_0 \varphi_0\left(t+\tau+\frac{\pi}{2}+\beta\pi\right) dt \varphi_0(u+\tau)\,du \\ &= (2\pi)^{-\frac{1}{q'}} \int_T \varepsilon_0 \varphi_\alpha\left(u+\tau+\frac{\pi}{2}+\beta\pi\right) \varphi_0(u+\tau)\,du. \end{aligned}$$

For each function $f \in L^{2n} \cap W_{\infty}^{\alpha}$, we use the representation

$$f(t) = \frac{1}{2\pi} \int_T f(\tau)\,d\tau + (B_\alpha * f^{(\alpha)})(t)$$

and also the fact that the mean value of the function φ_0 over the period is equal to zero, we have

$$\begin{aligned} \int_T f(t) g_{q,\tau}(t)\,dt &= (2\pi)^{-\frac{1}{q'}} \int_T \int_T B_\alpha(t-u) f^{(\alpha)}(u)\,du e^{-i\alpha\pi} \varepsilon_0 \varphi_0\left(t+\tau+\frac{\pi}{2}+\beta\pi\right) dt \\ &= (2\pi)^{-\frac{1}{q'}} \int_T \int_T B_\alpha(u-t) \varepsilon_0 \varphi_0\left(t+\tau+\frac{\pi}{2}+\beta\pi\right) dt f^{(\alpha)}(u)\,du \\ &= (2\pi)^{-\frac{1}{q'}} \int_T \varepsilon_0 \varphi_\alpha\left(u+\tau+\frac{\pi}{2}+\beta\pi\right) f^{(\alpha)}(u)\,du. \end{aligned}$$

In that case

$$\int_T \varphi_{\alpha,\tau}(t) g_{q,\tau}(t)\,dt - \int_T f(t) g_{q,\tau}(t)\,dt$$
$$= (2\pi)^{-\frac{1}{q'}} \int_T [\varphi_0(t+\tau) - f^{(\alpha)}(t)]\varepsilon_0 \varphi_\alpha\left(t+\tau+\frac{\pi}{2}+\beta\pi\right) dt. \tag{5.3}$$

Since $\|f^{(\alpha)}\|_\infty \leqslant 1$ and $\|\varphi_0\|_\infty = 1$, for each τ and almost all $t \in T$, at least, one of the relations $\operatorname{sgn}(\varphi_0(t+\tau) - f^{(\alpha)}(t)) = \operatorname{sgn}\varphi_0(t+\tau)$ or $\operatorname{sgn}(\varphi_0(t+\tau) - f^{(\alpha)}(t)) = 0$ is valid. In addition, from [24,25], we know that $\varphi'_\alpha(t) = H(t)$. Hence, $\operatorname{sgn}\varphi_\alpha(t) = \operatorname{sgn} H(t-\frac{\pi}{2}) = \varepsilon_0 \operatorname{sgn}\sin(t-\frac{\pi}{2}-\beta\pi)$, there holds $\varepsilon_0 \operatorname{sgn}\varphi_\alpha(t+\frac{\pi}{2}+\beta\pi) = \operatorname{sgn}\,\sin t = \operatorname{sgn}\varphi_0(t)$. So, for almost all $t \in T$, the following relation is valid:

$$(\varphi_0(t+\tau) - f^{(\alpha)}(t))\varepsilon_0\varphi_\alpha\left(t+\tau+\frac{\pi}{2}+\beta\pi\right)$$
$$= |(\varphi_0(t+\tau) - f^{(\alpha)}(t))| \cdot \left|\varepsilon_0\varphi_\alpha\left(t+\tau+\frac{\pi}{2}+\beta\pi\right)\right|.$$

Therefore, we get

$$\int_T (\varphi_0(t+\tau) - f^{(\alpha)}(t))\varepsilon_0\varphi_\alpha\left(t+\tau+\frac{\pi}{2}+\beta\pi\right) dt$$
$$= \int_T |(\varphi_0(t+\tau) - f^{(\alpha)}(t))| \cdot \left|\varepsilon_0\varphi_\alpha\left(t+\tau+\frac{\pi}{2}+\beta\pi\right)\right| dt. \tag{5.4}$$

Hence from relations (5.2)–(5.4) we derive the inequality

$$E(\varphi_{\alpha,\tau}, L^{2n} \cap W^\alpha_\infty)_q$$
$$\geqslant (2\pi)^{-\frac{1}{q'}} \inf_{f \in L^{2n}\cap W^\alpha_\infty} \int_T |(\varphi_0(t+\tau) - f^{(\alpha)}(t))| \cdot \left|\varepsilon_0\varphi_\alpha\left(t+\tau+\frac{\pi}{2}+\beta\pi\right)\right| dt. \tag{5.5}$$

Let us now prove the existence of a number τ_0 and of an absolute constant $c_0 > 0$ such that for any $f \in L^{2n} \cap W^\alpha_\infty$ the following inequality is valid:

$$\int_T |\varphi_0(t+\tau_0) - f^{(\alpha)}(t)|\,dt \geqslant c_0 n^{-1}. \tag{5.6}$$

Suppose that Y_L is the subspace of functions f from L^{2n} such that $f^{(\alpha)}$ exists as an element in the space L_∞ and $\|f^{(\alpha)}\|_\infty < \infty$, and suppose that Y^α_L denote the linearized subspace of the set $\{f^{(\alpha)},\ f \in Y_L\}$.

Also note that for any $\tau \in T$ and $f \in L^{2n} \cap W^\alpha_\infty$, taking into account the inequality $\|\varphi_{0,\tau} - f^{(\alpha)}\|_\infty \leqslant 2$, we can obtain

$$\int_T |\varphi_{0,\tau}(t) - f^{(\alpha)}(t)|\,dt \geqslant \frac{1}{2}\int_T |\varphi_{0,\tau}(t) - f^{(\alpha)}(t)|^2\,dt.$$

From this inequality, it follows that the curve $k(\varphi_0)$ generated by translations of the function φ_0 satisfies the inequalities

$$E(k(\varphi_0), L^{2n} \cap W_\infty^\alpha)_1 \geqslant \tfrac{1}{2}(E(k(\varphi_0), Y_L^\alpha)_2)^2 \geqslant \tfrac{1}{2}(d_{2n}(k(\varphi_0), L_2(T)))^2. \tag{5.7}$$

Since

$$\varphi_0(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}(2k-1)^{-1}\sin(2k-1)t,$$

using Lemma 2, we get

$$d_{2n}(k(\varphi_0), L_2(T)) = \frac{4}{\pi}\left(\pi\sum_{k=n+1}^{\infty}(2k-1)^{-2}\right)^{\frac{1}{2}} \geqslant cn^{-\frac{1}{2}}, \tag{5.8}$$

where c is an absolute constant.

Thus, from (5.7) and (5.8), we see that the estimate

$$E(k(\varphi_0), L^{2n} \cap W_\infty^\alpha)_1 \geqslant c_0 n^{-1},$$

where c_0 is an absolute constant. This estimate also implies the existence of a number τ_0 for which inequality (5.6) holds.

For fixed τ_0 and c_0, we assume $x(t) := |\varphi_0(t) - f^{(\alpha)}(t-\tau_0)|$ and $y(t) := |\varphi_\alpha(t + \frac{\pi}{2} + \beta\pi)|$. Obviously, by (5.6), $0 \leqslant x(t) \leqslant 2,\ t \in T$, and $\|x\|_1 \geqslant c_0 n^{-1}$. Here $y(t) \in C(T)$ and $y(t) \geqslant 0,\ t \in T$.

It is easy to see that

$$\int_T |\varphi_0(t+\tau_0) - f^{(\alpha)}(t)| \cdot \left|\varphi_\alpha\left(t + \tau_0 + \frac{\pi}{2} + \beta\pi\right)\right| dt = \int_T x(t)y(t)\,dt. \tag{5.9}$$

To obtain a lower bound for the last integral, we use Lemma 3 with $A = 2$, and $B = c_0 n^{-1}$. It is readily verified that the corresponding set $D(A, B)$ consists of three closed intervals: $[0, \delta]$, $[\pi - \delta, \pi + \delta]$, and $[2\pi - \delta, 2\pi]$ such that $c_1 n^{-1} \leqslant \delta \leqslant c_2 n^{-1}$, where c_1 and c_2 are independent of n and τ_0.

Since the points 0, π, and 2π are zeros of the function $y(t) = |\varphi_\alpha(t + \frac{\pi}{2} + \beta\pi)|$ (see [24,25]), we have the inequality

$$\int_{D(A,B)} y(t)\,dt \geqslant cn^{-2},$$

where c is independent of n and τ_0. Thus

$$\int_T x(t)y(t)\,dt \geqslant cn^{-2}.$$

Taking into account relation (5.9), we get the following inequality

$$\int_T |\varphi_0(t+\tau_0) - f^{(\alpha)}(t)| \cdot |\varphi_\alpha\left(t + \tau_0 + \frac{\pi}{2} + \beta\pi\right)|\,dt \geqslant cn^{-2}.$$

Substituting this estimate into (5.5), we find that

$$E(\varphi_{\alpha,\tau_0}, L^{2n} \cap W^\alpha_\infty)_q \geqslant cn^{-2},$$

where $c = c(\alpha, q)$ is independent of τ_0 and of the subspace L^{2n}.

But in that case

$$E(k(\varphi_\alpha), L^{2n} \cap W^\alpha_\infty)_q \geqslant cn^{-2},$$

for any subspace L^{2n}. Since the choice of L^{2n} is arbitrary, the last inequality yields inequality (5.1), and from (5.1) we obtain the inequality

$$K_n(W^\alpha_\infty, W^\alpha_\infty)_q \gg n^{-2}, \quad n \in \mathbb{Z}_+.$$

Thus, the lower bound in (1.2) is proved, and hence the proof of Theorem 2 is complete.

From the proof of Theorem 2, we get the following corollary.

Corollary 1. *If* $0 < \alpha \leqslant 2$, *and* $1 \leqslant q \leqslant \infty$, *then*

$$n^{-\alpha} \ll d_n(W^\alpha_\infty, L_q(T)) \leqslant K_n(W^\alpha_\infty, W^\alpha_\infty)_q \ll n^{-\alpha}, \quad n \in \mathbb{Z}_+.$$

6. Proof of Theorem 3

6.1. Upper estimate

We can obtain the desired upper estimate by using the Jackson operators J_n defined as in (4.1). If $f \in W^\alpha_1$, $\alpha > 0$, then it is easy to prove that $J_n(f, \cdot) \in W^\alpha_1$. Therefore, it is sufficient to estimate the deviation

$$\|f - J_n(f, \cdot)\|_q \ll n^{-\min\{\alpha - 1 + \frac{1}{q}, 2\}},$$

for each $f \in W^\alpha_1$, and $\alpha > 1 - \frac{1}{q}$ $(1 \leqslant q \leqslant \infty)$. Here the constant C only depends on α and q.

Using the generalized Minkowski integral inequality, we have

$$\begin{aligned}
\|f - J_n(f, \cdot)\|_q &= \left\| \int_T [f(\cdot) - f(\cdot + t)] K_n(t)\, dt \right\|_q \\
&= \left\| \int_0^\pi [\Delta_t^2 f(\cdot + t)] K_n(t)\, dt \right\|_q \leqslant \int_0^\pi \|\Delta_t^2 f(\cdot + t)\|_q K_n(t)\, dt \\
&= \int_0^\pi \left(\int_T \left| \int_T [\Delta_t^2 B(x - \tau + t)] f^{(\alpha)}(\tau)\, d\tau \right|^q dx \right)^{\frac{1}{q}} K_n(t)\, dt \\
&\leqslant w_2\left(B_\alpha, \frac{1}{n}\right)_q \int_0^\pi (nt + 1)^2 K_n(t)\, dt \ll w_2\left(B_\alpha, \frac{1}{n}\right)_q. \qquad (6.1)
\end{aligned}$$

It remains to prove the upper estimate of $w_2(B_\alpha, \frac{1}{n})_q$.

For $0 < \alpha < 1$, from [31], we know

$$\Gamma(\alpha) B_\alpha(x) = \lim_{n \to \infty} \left\{ x^{\alpha - 1} + (x + 2\pi)^{\alpha - 1} + \cdots + (x + 2\pi n)^{\alpha - 1} - (2\pi)^{\alpha - 1} \frac{n^\alpha}{\alpha} \right\},$$

for $0 < x < 2\pi$, $B_\alpha(x)$ is periodic, and

$$B_\alpha(x) = \psi_\alpha(x) + \gamma_\alpha(x), \quad -2\pi < x < 2\pi, \ \ 0 < \alpha < 1,$$

where $\psi_\alpha(x) = \frac{1}{\Gamma(\alpha)} x_+^{\alpha-1}$ is the function equal to $\frac{1}{\Gamma(\alpha)} x^{\alpha-1}$ for $x > 0$ and to 0 for $x < 0$, $\gamma_\alpha(x)$ is a function with derivatives of all orders for $x > -2\pi$. Furthermore, we also see that the inequalities

$$|B_\alpha(x)| \leqslant C_\alpha |x|^{\alpha-1}, \quad |B_\alpha'(x)| \leqslant C_\alpha |x|^{\alpha-2}, \quad |B_\alpha''(x)| \leqslant C_\alpha |x|^{\alpha-3} \tag{6.2}$$

valid for $0 < |x| < \pi$, and C_α depends on α only.

Consequently, write $h = \frac{1}{n}$, then

$$w_2(B_\alpha, h)_q = \left(\int_T |\Delta_h^2 B_\alpha(x)|^q \, dx \right)^{\frac{1}{q}} = \int_{|x| \leqslant 2h} + \int_{2h \leqslant |x| \leqslant \pi} = A + B.$$

It is easy to see that

$$|A| \leqslant 4 \left(\int_{-3h}^{3h} |B_\alpha(x)|^q \, dx \right)^{\frac{1}{q}} \leqslant C \left(\int_0^{3h} x^{(\alpha-1)q} \, dx \right)^{\frac{1}{q}} \leqslant C h^{\alpha - 1 + \frac{1}{q}}.$$

By the mean-value theorem and the third inequality of (6.2), we can deduce

$$\begin{aligned} |B| &= h^2 \left(\int_{2h \leqslant |x| \leqslant \pi} |B_\alpha''(x + \theta h)|^q \, dx \right)^{\frac{1}{q}} \leqslant h^2 \left(\int_{2h \leqslant |x| \leqslant \pi} (|x| - h)^{(\alpha-3)q} \, dx \right)^{\frac{1}{q}} \\ &\leqslant C h^2 \left(\int_{2h}^{\infty} x^{(\alpha-3)q} \, dx \right)^{\frac{1}{q}} \leqslant C h^{\alpha - 1 + \frac{1}{q}}. \end{aligned}$$

Hence, we have

$$w_2 \left(B_\alpha, \frac{1}{n} \right)_q \ll n^{-(\alpha - 1 + \frac{1}{q})} \quad \text{for} \quad 1 - \frac{1}{q} < \alpha < 1.$$

For $\alpha = 1$, since

$$B_1(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}}{}' \frac{e^{ikx}}{ik} = \frac{\pi - x}{2\pi}, \quad 0 < x < 2\pi,$$

we have

$$\begin{aligned} w_1(B_1, h)_q &= \left(\int_0^{2\pi} |B_1(x+h) - B_1(x)|^q \, dx \right)^{\frac{1}{q}} \\ &= \left(\int_0^{2\pi - h} \left| \frac{\pi - x - h}{2\pi} - \frac{\pi - x}{2\pi} \right|^q dx + \int_{2\pi - h}^{2\pi} \left| \frac{3\pi - x - h}{2\pi} - \frac{\pi - x}{2\pi} \right|^q dx \right)^{\frac{1}{q}} \\ &= \left(\left| \frac{h}{2\pi} \right|^q \cdot (2\pi - h) + \left| \frac{2\pi - h}{2\pi} \right|^q \cdot h \right)^{\frac{1}{q}} \leqslant C h^{\frac{1}{q}}, \end{aligned}$$

that is

$$w_2\left(B_1, \frac{1}{n}\right)_q \leqslant 2w_1\left(B_1, \frac{1}{n}\right)_q \ll n^{-\frac{1}{q}}.$$

For $1 < \alpha < 2$, set $\lambda = \alpha - 1$. Suppose that c is one zero of $B_\alpha(x)$ on $[0, 2\pi]$, then

$$B_\alpha(x) = B_{\lambda+1}(x) = \int_c^x B_\lambda(t)\,dt = \frac{t^\lambda}{\Gamma(\lambda)\lambda}\bigg|_c^x + \int_c^x \gamma_\lambda(t)\,dt$$

$$= \frac{x^\lambda}{\Gamma(\lambda+1)} - \frac{c^\lambda}{\Gamma(\lambda+1)} + \int_c^x \gamma_\lambda(t)\,dt = \frac{x^\lambda}{\Gamma(\lambda+1)} + \gamma_{\lambda+1}(x).$$

Further, we have

$$\begin{aligned}
w_2(B_\alpha, h)_q &= \left(\int_T |B_\alpha(x+h) + B_\alpha(x-h) - 2B_\alpha(x)|^q\,dx\right)^{\frac{1}{q}} \\
&\leqslant \frac{1}{\Gamma(\lambda+1)}\left(\int_T |(x+h)_+^\lambda + (x-h)_+^\lambda - 2x_+^\lambda|^q\,dx\right)^{\frac{1}{q}} \\
&\quad + \left(\int_T |\gamma_{\lambda+1}(x+h) + \gamma_{\lambda+1}(x-h) - 2\gamma_{\lambda+1}(x)|^q\,dx\right)^{\frac{1}{q}} \\
&= \frac{1}{\Gamma(\lambda+1)}\left(\int_{|x|\leqslant 2h} |(x+h)_+^\lambda + (x-h)_+^\lambda - 2x_+^\lambda|^q\,dx\right)^{\frac{1}{q}} \\
&\quad + \frac{1}{\Gamma(\lambda+1)}\left(\int_{2h\leqslant|x|\leqslant\pi} |(x+h)_+^\lambda + (x-h)_+^\lambda - 2x_+^\lambda|^q\,dx\right)^{\frac{1}{q}} \\
&\quad + \left(\int_T \left|\int_0^h\int_0^h \gamma''_{\lambda+1}(x+t_1+t_2)\,dt_1\,dt_2\right|^q dx\right)^{\frac{1}{q}} \\
&= A + B + C.
\end{aligned}$$

It is easy to see that for $0 < \lambda < 1$ there hold the following estimates:

$$A \leqslant \frac{4}{\Gamma(\lambda+1)}\left(\int_{-3h}^{3h} |x|^{\lambda q}\,dx\right)^{\frac{1}{q}} = \frac{8}{\Gamma(\lambda+1)}\left(\int_0^{3h} x^{\lambda q}\,dx\right)^{\frac{1}{q}} \leqslant Ch^{\lambda+\frac{1}{q}},$$

$$\begin{aligned}
B &= \frac{1}{\Gamma(\lambda+1)}h^2\left(\int_{2h\leqslant|x|\leqslant\pi} (x+\theta h)^{(\lambda-2)q}\,dx\right)^{\frac{1}{q}} \\
&\leqslant Ch^2\left(\int_{2h\leqslant|x|\leqslant\pi} (|x|-h)^{(\lambda-2)q}\,dx\right)^{\frac{1}{q}} \\
&\leqslant Ch^2\left(\int_{2h}^{\infty} x^{(\lambda-2)q}\,dx\right)^{\frac{1}{q}} \leqslant Ch^{\lambda+\frac{1}{q}}.
\end{aligned}$$

By the Minkowskii inequality, we obtain

$$
\begin{aligned}
C &\leqslant \int_0^h \int_0^h \left(\int_T |\gamma''_{\beta+1}(x+t_1+t_2)|^q \, dx \right)^{\frac{1}{q}} dt_1 \, dt_2 \\
&\leqslant \|\gamma''_{\beta+1}\|_{L_{q[-\pi,3\pi]}} h^2.
\end{aligned}
$$

Consequently, $A + B + C = O(h^{\lambda+\frac{1}{q}})$, that is

$$
w_2\left(B_\alpha, \frac{1}{n}\right)_q \ll n^{-(\alpha-1+\frac{1}{q})} \quad \text{for } 1 < \alpha < 2.
$$

For $\alpha = 2$, by the property of modulus of continuity, we have

$$
w_2\left(B_2, \frac{1}{n}\right)_q \leqslant \frac{1}{n} w\left(B_1, \frac{1}{n}\right)_q \ll n^{-(1+\frac{1}{q})}.
$$

For $\alpha > 2$, using the method similar to the case $1 < \alpha < 2$, we can deduce

$$
w_2\left(B_\alpha, \frac{1}{n}\right)_q \ll n^{-(\alpha-1+\frac{1}{q})}, \quad 2 < \alpha < 3 - \frac{1}{q},
$$

$$
w_2(B_\alpha, \frac{1}{n})_q \ll n^{-2}, \quad \alpha \geqslant 3 - \frac{1}{q}.
$$

From above all, we get

$$
w_2\left(B_\alpha, \frac{1}{n}\right)_q \ll n^{-\min\{\alpha-1+\frac{1}{q},\, 2\}}, \quad \alpha \geqslant 1 - \frac{1}{q}.
$$

Up to now, we complete the upper estimate of the asymptotic formula in Theorem 3.

6.2. *Lower estimate*

As above, let $\varphi_0(t) := \operatorname{sgn} \sin t$, and $\psi_0(\cdot) := \frac{1}{4}\varphi_0(\cdot)$ the function whose total variation on the closed interval $[0, 2\pi]$ is equal to 1. Let the function $\psi_{\alpha-1}(\cdot) := (B_{\alpha-1} * \psi_0)(\cdot)$, and

$$
k(\psi_{\alpha-1}) := \{\psi_{\alpha-1,\tau} | \psi_{\alpha-1,\tau}(\cdot) := \psi_{\alpha-1}(\cdot + \tau), \tau \in [0, 2\pi]\}
$$

be the curve generated by translations of $\psi_{\alpha-1}(\cdot)$.

Next, we will prove that if $\alpha \geqslant 3 - \frac{1}{q}$, then

$$
K_n(k(\psi_{\alpha-1}), W_1^\alpha)_q \gg n^{-2}, \quad n \in \mathbb{Z}_+. \tag{6.3}
$$

For a fixed arbitrary subspace $L^{2n} \subset L_q$ of dimension $\leqslant 2n$, by Lemma 1, we find that for each fixed $\tau \in [0, 2\pi]$ the function $\psi_{\alpha-1,\tau}(\cdot) := \psi_{\alpha-1}(\cdot + \tau)$ satisfies the inequality

$$
E(\psi_{\alpha-1,\tau}, L^{2n} \cap W_1^\alpha)_q d = \sup_{\substack{g \in L_{q'} \\ \|g\|_{q'} \leqslant 1}} \left(\int_T \psi_{\alpha-1,\tau}(t) g(t) \, dt - \sup_{f \in L^{2n} \cap W_1^\alpha} \int_T f(t) g(t) \, dt \right), \tag{6.4}
$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

If $\alpha \geqslant 3 - \dfrac{1}{q}$, then we choose the function

$$g_{q,\tau}(t) := (2\pi)^{-\frac{1}{q'}} e^{-i(\alpha-1)\pi} \varepsilon_0 \varphi_0 \left(t + \tau + \frac{\pi}{2} + \beta\pi \right), \tag{6.5}$$

here β satisfies $0 \leqslant \beta < 1$, $H(\beta\pi) = 0$, while

$$H(t) = \sum_{v=0}^{\infty} \frac{\cos[(2v+1)t - \frac{\pi(\alpha-1)}{2}]}{(2v+1)^{(\alpha-1)}}.$$

From [24,25], it is seen that $\operatorname{sgn} H(t) = \varepsilon_0 \operatorname{sgn} \sin(t - \beta\pi)$, $\varepsilon_0 = \pm 1$.

Noticing that $g_{q,\tau} \in L_{q'}$ and $\|g_{q,\tau}\|_{q'} = 1$. Therefore, we obtain the inequality

$$\begin{aligned}
&E(\psi_{\alpha-1,\tau}, L^{2n} \cap W_1^\alpha)_q \\
&\quad \geqslant \int_T \psi_{\alpha-1,\tau}(t) g_{q,\tau}(t)\, dt - \sup_{f \in L^{2n} \cap W_1^\alpha} \int_T f(t) g_{q,\tau}(t)\, dt \\
&\quad = \inf_{f \in L^{2n} \cap W_1^\alpha} \left(\int_T \psi_{\alpha-1,\tau}(t) g_{q,\tau}(t)\, dt - \int_T f(t) g_{q,\tau}(t)\, dt \right).
\end{aligned} \tag{6.6}$$

Taking into account definition (6.5) for $g_{q,\tau}$, similar to the proof of (5.3), we obtain the following relation

$$\begin{aligned}
&\int_T \psi_{\alpha-1,\tau}(t) g_{q,\tau}(t)\, dt - \int_T f(t) g_{q,\tau}(t)\, dt \\
&\quad = (2\pi)^{-\frac{1}{q'}} \int_T [\psi_0(t+\tau) - f^{(\alpha-1)}(t)] \varepsilon_0 \varphi_{\alpha-1} \left(t + \tau + \frac{\pi}{2} + \beta\pi \right) dt.
\end{aligned} \tag{6.7}$$

Since for any $f \in L^{2n} \cap W_1^\alpha$ the total variation of the derivative $f^{(\alpha-1)}$ on the closed interval T is at most 1 and the mean value over the period of the functions $\varphi_{\alpha-1}$ is equal to 0, it follows that in relation (6.7) we can assume that $\|f^{(\alpha-1)}\|_\infty \leqslant \frac{1}{4}$. In this case, for almost all $t \in T$ the following relation is valid:

$$\begin{aligned}
&[\psi_0(t+\tau) - f^{(\alpha-1)}(t)] \varepsilon_0 \varphi_{\alpha-1} \left(t + \tau + \frac{\pi}{2} + \beta\pi \right) \\
&\quad = |\psi_0(t+\tau) - f^{(\alpha-1)}(t)| \cdot \left| \varphi_{\alpha-1} \left(t + \tau + \frac{\pi}{2} + \beta\pi \right) \right|.
\end{aligned} \tag{6.8}$$

Using Lemma 2, we can prove the existence of a number τ_0 and of an absolute constant $c_0 > 0$ such that for any $f \in L^{2n} \cap W_1^\alpha$ the inequality is valid:

$$\int_T |\psi_0(t+\tau_0) - f^{(\alpha-1)}(t)|\, dt \geqslant c_0 n^{-1}.$$

Next, we assume $x(t) := |\psi_0(t) - f^{(\alpha-1)}(t-\tau_0)|$ and $y(t) := |\varphi_{\alpha-1}(t + \frac{\pi}{2} + \beta\pi)|$. Then from relations (6.6)–(6.8) we obtain

$$E(\psi_{\alpha-1,\tau_0}, L^{2n} \cap W_1^\alpha)_q \geqslant (2\pi)^{-\frac{1}{q'}} \int_0^{2\pi} x(t) y(t)\, dt, \tag{6.9}$$

in which $0 \leqslant x(t) \leqslant \frac{1}{2}$ and $\|x\|_1 \geqslant c_0 n^{-1}$, while $y(t) \in C(T)$ and $y(t) \geqslant 0$.

To obtain a lower bound for the integral in (6.9), we apply Lemma 3 with $A = \frac{1}{2}$ and $B = c_0 n^{-1}$. It is readily verified that in this case the subset $D(A, B)$ of the interval $[0, 2\pi]$ will also consist of three closed interval: $[0, \delta]$, $[\pi - \delta, \pi + \delta]$, and $[2\pi - \delta, 2\pi]$ such that $c_1 n^{-1} \leqslant \delta \leqslant c_2 n^{-1}$, where c_1 and c_2 are independent of n and τ_0. Since the points $0, \pi$, and 2π are zeros of the function $y(t) = |\varphi_{\alpha-1}(t + \frac{\pi}{2} + \beta\pi)|$, we have

$$\int_{D(A,B)} y(t)\,dt \geqslant cn^{-2},$$

where c is independent of n and τ_0. Thus

$$\int_0^{2\pi} x(t)y(t)\,dt \geqslant cn^{-2}.$$

In this case, from (6.9), we deduce the inequality

$$E(\psi_{\alpha-1,\tau_0}, L^{2n} \cap W_1^\alpha)_q \gg n^{-2}, \tag{6.10}$$

where $c = c(\alpha, q)$ is independent of n and τ_0. Since the choice of L^{2n} is arbitrary, from this estimate (6.10), we obtain inequality (6.3).

Obviously, the set $k(\psi_{\alpha-1})$ does not belong to the class W_1^α. However, if we replace the function $\psi_{\alpha-1}$ by its Steklov function $\psi_{\varepsilon,\alpha-1}$ and denote by $k(\psi_{\varepsilon,\alpha-1})$ the curves generated by translations of $\psi_{\varepsilon,\alpha-1}(\cdot)$, then $k(\psi_{\varepsilon,\alpha-1}) \subset W_1^\alpha$ and

$$\|k(\psi_{\alpha-1}) - k(\psi_{\varepsilon,\alpha-1})\|_q \to 0, \quad \varepsilon \to 0,$$

for $1 \leqslant q \leqslant \infty$. Taking this into account, from (6.3) we easily obtain the desired lower bound in (1.4) for $\alpha \geqslant 3 - \frac{1}{q}$.

If $1 - \frac{1}{q} < \alpha < 3 - \frac{1}{q}$, and $1 \leqslant q \leqslant 2$, then we will prove that

$$K_n(W_1^\alpha, W_1^\alpha)_q \gg n^{-(\alpha-1+\frac{1}{q})}. \tag{6.11}$$

In the view of $K_n(W_1^\alpha, W_1^\alpha)_q \geqslant d_n(W_1^\alpha, L_q(T))$. It is sufficient to prove that

$$d_n(W_1^\alpha, L_q(T)) \gg n^{-(\alpha-1+\frac{1}{q})}, \tag{6.12}$$

for all $\alpha > 0$, and $1 \leqslant q \leqslant 2$.

Next, we still use the notations as in the section Section 5. Similarly, from [19], we may see that

$$\begin{aligned} d_n(W_1^\alpha, L_q(T)) &= \inf_{X_n} \sup_{\|f^{(\alpha)}\|_1 \leqslant 1} \inf_{g \in X_n} \|f - g\|_q \\ &= \inf_{X_n} \sup_{\|f^{(\alpha)}\|_1 \leqslant 1} \sup_{\substack{h \perp X_n \\ \|h\|_{q'} \leqslant 1}} (f, h). \end{aligned}$$

Here $\frac{1}{q} + \frac{1}{q'} = 1$, X_n is taken over all subspaces of $L_q(T)$ and $h \perp X_n$ means that $(g, h) = \int_0^{2\pi} g(x)h(x)\,dx = 0$ for all $g \in X_n$.

Set $Y_M = \mathrm{span}(\phi_1, \ldots, \phi_M)$. Thus, we have

$$d_n(W_1^\alpha, L_q(T)) \geqslant \inf_{X_n} \sup_{\substack{\|f^{(\alpha)}\|_1 \leqslant 1 \\ f \in Y_M}} \sup_{\substack{h \perp X_n \\ \|h\|_{q'} \leqslant 1 \\ h \in Y_M}} (f, h).$$

Let $f=\sum_{k=1}^{M} a_k\phi_k$ and $h=\sum_{k=1}^{M} b_k\phi_k$. Then we have

$$\|f^{(\alpha)}\|_1 \leqslant CM^{\alpha-1}\|a\|_1\|\phi\|_1^{1-\frac{\alpha}{1+[\alpha]}}\|\phi^{(1+[\alpha])}\|_1^{\frac{\alpha}{1+[\alpha]}},$$

$$\|h\|_{q'} = M^{-\frac{1}{q'}}\|\phi\|_{q'}\|b\|_{q'},$$

and

$$(f,h)=\sum_{k=1}^{M} a_k b_k\|\phi_k\|_2^2 = M^{-1}\|\phi\|_2^2\sum_{k=1}^{M} a_k b_k.$$

Assume $X_n=\operatorname{span}\{g_1,\ldots,g_n\}$ and define $d_{i,j}=(g_i,\phi_j),\ \ i=1,\ldots,n;\ j=1,\ldots,M$. Let $d^i,\ \ i=1,\ldots,n$, denote the ith row of the matrix $D=(d_{i,j})$. Then there holds the following estimate:

$$\begin{aligned} d_n(W_1^\alpha, L_q(T)) &\geqslant M^{-(\alpha-1+\frac{1}{q})}\inf_{d^1,\ldots,d^n}\ \sup_{\|a\|_1\leqslant 1}\ \sup_{\substack{(d^i,b)=0,i=1,\ldots,n\\ \|b\|_{q'}\leqslant 1}} (a,b) \\ &\geqslant M^{-(\alpha-1+\frac{1}{q})}\inf_{d^1,\ldots,d^n}\ \sup_{\|a\|_1\leqslant 1}\ \inf_{\alpha_1,\ldots,\alpha_n}\left\| a-\sum_{i=1}^{n}\alpha_i d^i\right\|_q \\ &\geqslant M^{-(\alpha-1+\frac{1}{q})} d_n(\mathcal{F}_1, l_q^M), \end{aligned}$$

where $\mathcal{F}_1=\{(a_1,\ldots,a_n):\sum_{j=1}^{M}|a_j|\leqslant 1\}$.

Taking $M=2n$, from Pinkus [19, Chapter VI, Theorem 2.2], there holds

$$d_n(\mathcal{F}_1, l_q^M)\geqslant 2^{-\frac{1}{2}},$$

for $1\leqslant q\leqslant 2$. Then we get

$$d_n(W_1^\alpha, L_q(T))\gg n^{-(\alpha-1+\frac{1}{q})},$$

which is (6.12).

Sum up, we finally complete the proof of Theorem 3.

From the proof of Theorem 3, we have the following corollary.

Corollary 2. *Let* $1\leqslant q\leqslant 2$. *Then*

$$d_n(W_1^\alpha, L_q(T))\gg n^{-(\alpha-1+\frac{1}{q})},\ \alpha>1-\frac{1}{q};$$

$$n^{-(\alpha-1+\frac{1}{q})}\gg K_n(W_1^\alpha, W_1^\alpha)_q\geqslant d_n(W_1^\alpha, L_q(T))\gg n^{-(\alpha-1+\frac{1}{q})},\ 3-\frac{1}{q}\geqslant\alpha>1-\frac{1}{q}.$$

Remark. In Theorem 3, the asymptotic estimate is open for the case $\alpha>0$, and $2\leqslant q\leqslant\infty$.

Acknowledgments

The authors would like to thank the two referees for their valuable comments and helpful suggestions. The authors also would like to thank Professor V.V. Arestov and Professor V.N. Konovalov for their kindly help when the authors looked for the related papers.

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